W57. In all triangle ABC holds

$$\sum \sin^2\frac{A}{2}\cos^2 A \geq \frac{3\left(s^2-(2R+r)^2\right)}{8R^2}$$

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Since $\sum \cos^2 A = \dfrac{4Rr+6R^2+r^2-s^2}{2R^2}$ and $\sum \cos^3 A = \dfrac{(2R+r)^3-3s^2r}{4R^3}-1$

then $\sum \sin^2\frac{A}{2}\cos^2 A = \frac{1}{2}\sum(1-\cos A)\cos^2 A = \frac{1}{2}(\sum\cos^2 A - \sum\cos^3 A) =$

$\dfrac{1}{2}\left(\dfrac{4Rr+6R^2+r^2-s^2}{2R^2} - \dfrac{(2R+r)^3-3s^2r}{4R^3}+1\right) = \dfrac{(8R^3-4R^2r-4Rr^2-r^3)-(2R-3r)s^2}{8R^3}$

and $\sum \sin^2\frac{A}{2}\cos^2 A \geq \dfrac{3\left(s^2-(2R+r)^2\right)}{8R^2} \iff$

$\dfrac{(8R^3-4R^2r-4Rr^2-r^3)-(2R-3r)s^2}{8R^3} \geq \dfrac{3\left(s^2-(2R+r)^2\right)}{8R^2} \iff$

$(8R^3-4R^2r-4Rr^2-r^3)-(2R-3r)s^2 \geq 3R\left(s^2-(2R+r)^2\right) \iff$

$\left(3R(2R+r)^2-4Rr^2-4R^2r+8R^3-r^3\right) \geq (5R-3r)s^2 \iff$

(**1**) $20R^3+8R^2r-Rr^2-r^3 \geq (5R-3r)s^2$.

Since $s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}$ (Fundamental Geometric Inequality) and $R \geq 2r$ (Euler's Inequality) remains to prove inequality

$20R^3+8R^2r-Rr^2-r^3 \geq (5R-3r)\left(2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}\right) \iff$

$20R^3+8R^2r-Rr^2-r^3-(5R-3r)(2R^2+10Rr-r^2) \geq 2(5R-3r)(R-2r)\sqrt{R(R-2r)} \iff$

$10R^3-36R^2r+34Rr^2-4r^3 \geq 2(5R-3r)(R-2r)\sqrt{R(R-2r)} \iff$

(**2**) $5R^3-18R^2r+17Rr^2-2r^3 \geq (5R-3r)(R-2r)\sqrt{R(R-2r)}$.

Let $t := \frac{R}{r}$. Then $t \geq 2$ and (**2**)$\iff$ $5t^3-18t^2+17t-2 \geq (5t-3)(t-2)\sqrt{t(t-2)}$.

We have $5t^3-18t^2+17t-2-(5t-3)(t-2)\sqrt{t(t-2)} =$

$(t-2)(5t^2-8t+1)-(5t-3)(t-2)\sqrt{t(t-2)} = (t-2)\left(5t^2-8t+1-(5t-3)\sqrt{t(t-2)}\right) =$

$\dfrac{(t-2)(5t^2+2t+1)}{5t^2-8t+1+(5t-3)\sqrt{t(t-2)}} \geq 0$